Exact Localized and Periodic Solutions of the Ablowitz-Ladik Discrete Nonlinear Schrödinger System

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We have studied, analytically, the Ablowitz-Ladik discrete nonlinear Schrödinger system. We have found a set of exact solutions which includes as particular cases periodic solutions in terms of elliptic Jacobian functions, bright and dark soliton solutions, and quasi-periodic solutions. We have also found the range of parameters where each exact solution exists. – PACS: 02.30.Jr, 05.45.Yv, 42.65.Tg, 02.30.Gp.

Key words: Ablowitz-Ladik Discrete Nonlinear Schrödinger System; Discrete Soliton; Periodic Solution.

1. Introduction

Solitons have been encountered in many continuum systems. However, there are few discrete systems that support solitons. Discrete solitons in nonlinear lattices have been the focus of considerable attention in diverse branches of science [1]. Discrete solitons are possible in several physical settings, such as biological systems [2], atomic chains [3, 4], solid state physics [5], electrical lattices [6] and Bose-Einstein condensates [7]. Recently, the existence of discrete solitons in photonic structures (in arrays of coupled nonlinear optics waveguides [8 – 13] and in a nonlinear photonic crystal structure [14]) was announced and has attracted considerable attention in the scientific community. Photonic crystals, which are artificial microstructures having photonic bandgaps, can be used to precisely control propagation of optical pulses and beams. They are very useful for optical components such as waveguides, couplers, cavities and optical computers. It is possible to make discrete waveguides using photonic crystals. In this situation, 'discrete solitons' appear naturally and have interesting properties. Many scientists believe that the discrete solitons can have an important role in this technology.

The investigation of the soliton solutions of nonlinear equations play an important role in the study of nonlinear physical phenomena. For example, the wave phenomena observed in fluid dynamics, plasma and elastic media are often modelled by the bell shaped sech solutions and the kink shaped tanh travelling wave solutions. The exact solution, if available, of those nonlinear equations facilitates the verification of numerical solvers and aids in the stability analysis of solutions. Writing the soliton solutions of a nonlinear equation as the polynomials of hyperbolic functions, the equation can be changed into a nonlinear system of algebraic equations [15 – 20]. The system can be solved with the help of Maple. More recently, D. Baldwin et al. [21 – 23] presented an algorithm to find exact discrete soliton solutions in terms of tanh function of discrete nonlinear models, such as the Volterra lattice models [21], discrete Korteweg – de Vries (MKdV) equation [22], hybrid lattice [23], relativistic Toda lattice [21].

The integrable discrete nonlinear Schrödinger equation (IDNLS) is

$$\mathrm{i}\frac{\partial u_n}{\partial t}+\frac{u_{n-1}+u_{n+1}-2u_n}{h^2}+|u_n|^2(u_{n-1}+u_{n+1})=0, \tag{1}$$

where u_n are complex variables defined for all integer values of the site index n. The term $u_{n-1}+u_{n+1}-2u_n$ plainly approximates a second derivative term for a continuous system and so physically represents diffraction. The IDNLS is used in modeling diverse physical phenomena, for example sell trapping on a dimer [24], dynamics of an harmonic lattices [25] and pulse dynamics in nonlinear optics [26]. In fact, the

IDNLS can be viewed as a special case (letting $u_n = v_n^*$) of the Ablowitz-Ladik discrete nonlinear Schrödinger (AL) system

$$\mathrm{i}\frac{\partial u_n}{\partial t} + \frac{u_{n+1}+u_{n-1}-2u_n}{h^2} + u_n v_n(u_{n+1}+u_{n-1}) = 0,$$
$$-\mathrm{i}\frac{\partial v_n}{\partial t} + \frac{v_{n+1}+v_{n-1}-2v_n}{h^2} + u_n v_n(v_{n+1}+v_{n-1}) = 0. \tag{2}$$

The AL system has N soliton solutions and a rich mathematical structure. It first appeared in a somewhat implicit form in [27, 28].

In this paper, we develop a new algebraic method which further exceeds the applicability of the tanh method for the AL system with the help of Maple. We construct a series of travelling wave solutions including discrete solitons, quasi-periodic, Jacobian doubly periodic solutions.

2. Discrete Solitons and Quasi-periodic Solutions with Constant Phase

The nonlinear plane wave solutions of AL system (2) are defined by

$$u_{n+p_s} = \phi_{n+p_s}(\xi)\exp(\mathrm{i}\theta + \mathrm{i}\rho_s),$$
$$v_{n+p_s} = \psi_{n+p_s}(\xi)\exp(-\mathrm{i}\theta - \mathrm{i}\rho_s), \tag{3}$$

with

$$\xi \equiv \chi n + ct + \zeta, \quad \theta \equiv \kappa n + \omega t + \delta,$$
$$p_s = -1, 0, 1, \quad \rho_s = p_s\kappa. \tag{4}$$

The substitution of the expression (3) with (4) into system (2) and the separation of the real and imaginary parts lead to

$$[(\phi_{n+1}-\phi_{n-1})\psi_n\phi_n\sin(\kappa) + \frac{\partial\phi_n}{\partial\xi}c]h^2 + (\phi_{n+1}-\phi_{n-1})\sin(\kappa) = 0,$$
$$[(\phi_{n+1}+\phi_{n-1})\psi_n\phi_n\cos(\kappa) - \phi_n\omega]h^2 + (\phi_{n+1}+\phi_{n-1})\cos(\kappa) - 2\phi_n = 0,$$
$$[(\psi_{n+1}-\psi_{n-1})\psi_n\phi_n\sin(\kappa) + \frac{\partial\psi_n}{\partial\xi}c]h^2 + (\psi_{n+1}-\psi_{n-1})\sin(\kappa) = 0,$$
$$[(\psi_{n+1}+\psi_{n-1})\psi_n\phi_n\cos(\kappa) - \psi_n\omega]h^2 + (\psi_{n+1}+\psi_{n-1})\cos(\kappa) - 2\psi_n = 0. \tag{5}$$

Furthermore, we assume that

$$\phi_{n+p_s}(\xi) = \sum_{j=1}^{m} a_j\varphi^j(\xi + p_s\chi),$$
$$\psi_{n+p_s}(\xi) = \sum_{j=1}^{l} b_j\varphi^j(\xi + p_s\chi), \tag{6}$$

with φ satisfies the equation

$$\varphi' = \sqrt{d_0 + d_1\varphi + d_2\varphi^2 + d_3\varphi^3 + d_4\varphi^4},$$
$$\varphi \equiv \varphi(\xi + p_s\chi), \tag{7}$$

where the prime $'$ denotes $\mathrm{d}/\mathrm{d}\xi$, and d_j, $j = 0,1,2,3,4$, are arbitrary constants. Substituting the ansatz (6) into (5) and balancing the linear term of the highest order with the highest nonlinear term, we get $m = l = 1$. By considering the different values of d_0, d_1, d_2, d_3 and d_4, we have the following results.

2.1. Bright soliton

If $d_0 = d_1 = d_3 = 0$, (7) possesses solution

$$\varphi(\xi + p_s\chi) = \sqrt{-\frac{d_2}{d_4}}\mathrm{sech}[\sqrt{d_2}(\xi + p_s\chi)],$$
$$d_2 > 0, \quad d_4 < 0. \tag{8}$$

Substituting (6) with (8) into (5) and equating the coefficients of various powers of $\cosh(\sqrt{d_2}\xi)$ to zero we can get a set of algebraic systems for a_1, b_1, c and ω. Solving them gives

$$a_1 = \text{arbitrary constant},$$
$$b_1 = -\frac{d_4\sinh^2(\sqrt{d_2}\chi)}{h^2 d_2 a_1},$$
$$c = -\frac{2\sinh(\sqrt{d_2}\chi)\sin(\kappa)}{\sqrt{d_2}h^2},$$
$$\omega = \frac{-2 + 2\cos(\kappa)\cosh(\sqrt{d_2}\psi)}{h^2}. \tag{9}$$

Then we have the bright soliton solutions ($p_s = 0$)

$$u_n = a_1\sqrt{-\frac{d_2}{d_4}}\mathrm{sech}(\sqrt{d_2}\xi)\exp(\mathrm{i}\theta),$$
$$v_n = b_1\sqrt{-\frac{d_2}{d_4}}\mathrm{sech}(\sqrt{d_2}\xi)\exp(-\mathrm{i}\theta), \tag{10}$$

where the functions ξ and θ are defined by (4), and the parameters a_1, b_1, ω and c are given by (9). The

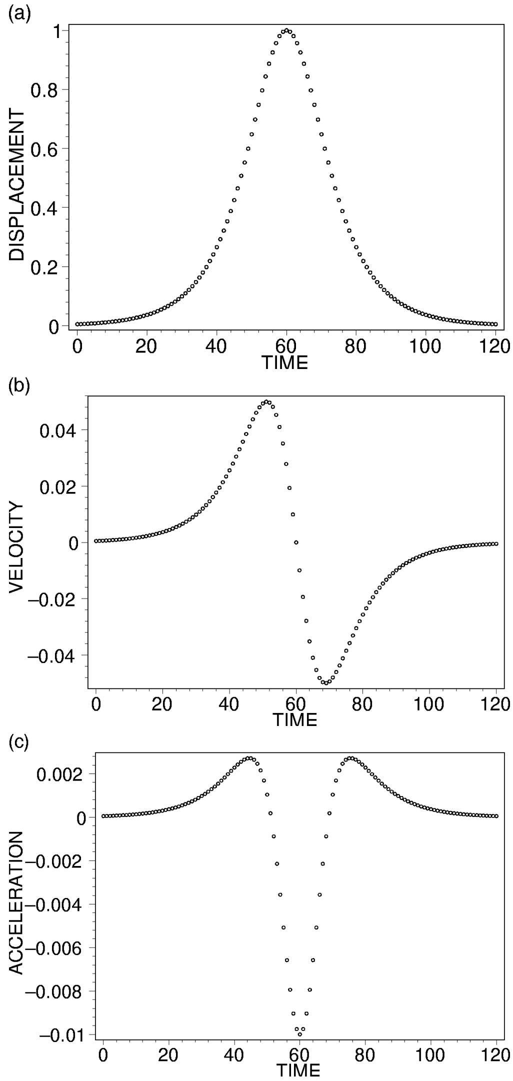

Fig. 1. Motion of discrete bright soliton for $|u_n|(|v_n|)$ with fixed n and parameters $d_2 = -d_4 = 1/100$, $a_1 = 1$, $h = -20$, $\chi = 10\text{arcsinh}(20)$, $\kappa = -\pi/2$, $\zeta = -60 - 10\text{arcsinh}(20)n$, where (a) displacement, (b) velocity, (c) acceleration.

amplitude and the width of the solitons are fixed and defined by the parameters of the ansatz. Figure 1 gives the motion of discrete bright soliton for $|u_n|(|v_n|)$ with fixed n. The constant $\zeta(\delta)$ in (4) is an arbitrary real constant, indicating translational invariance along the lattice. Although it seems simple, the translational invariance is not as trivial as in the case of the continuous equation. When $\zeta(\delta)$ is zero, the center of the soliton coincides with a lattice site. In this instance, the solution is symmetric. When $\zeta(\delta)$ is not zero, the soliton center is located between the lattice sites. Then the

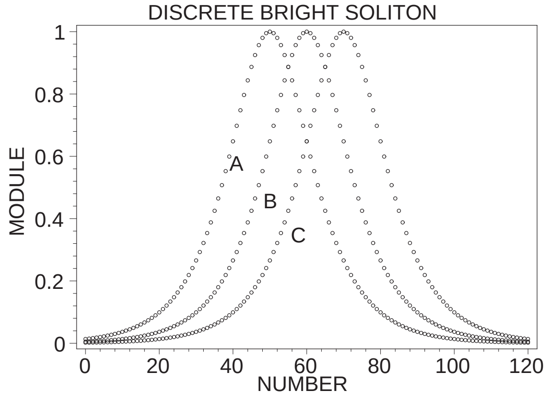

Fig. 2. Shape of discrete bright soliton for $|u_n|(|v_n|)$ with different phase ζ and parameters $d_2 = -d_4 = 1/100$, $a_1 = 1$, $h = \sinh(1/10)$, $\chi = 1$ at $t = 0$, where $\zeta = -50$ for A, $\zeta = -60$ for B, $\zeta = -70$ for C.

soliton shape is asymmetric. In this sense, the parameter $\zeta(\delta)$ produces a continuous family of solitons with variable shape, see Figure 2.

2.2. Oscillatory 'sec' Solution

When we consider a different range of (8) parameters $d_2 < 0$, $d_4 > 0$, we obtain solutions which give quasi-periodic oscillations; they do not actually repeat, unless the period of the solutions ($p_s = 0$) becomes commensurate with the period of the chain

$$\begin{aligned} u_n &= a_1\sqrt{-\frac{d_2}{d_4}}\sec(\sqrt{-d_2}\xi)\exp(\mathrm{i}\theta), \\ v_n &= b_1\sqrt{-\frac{d_2}{d_4}}\sec(\sqrt{-d_2}\xi)\exp(-\mathrm{i}\theta), \end{aligned} \tag{11}$$

where functions ξ and θ are defined by (4) with

$$\varphi = \sqrt{-\frac{d_2}{d_4}}\sec(\sqrt{-d_2}\xi), \quad d_2 < 0, \quad d_4 > 0,$$

$$b_1 = \text{arbitrary constant},$$

$$a_1 = \frac{d_4\sin^2(\sqrt{-d_2}\chi)}{h^2 d_2 b_1}, \quad c = -\frac{2\sin(\kappa)\sin(\sqrt{-d_2}\chi)}{h^2\sqrt{-d_2}},$$

$$\omega = \frac{-2+2\cos(\kappa)\cos(\sqrt{-d_2}\chi)}{h^2}.$$

In the continuous model a solution in terms of sec function would have singularities. In the discrete model the singularity may occur as well. However, this happens only when the solution becomes infinite at the site of the chain. This can be avoided by properly choosing the equation parameters and translation parameter $\zeta(\delta)$.

An example is $b_1 = -d_2 = d_4 = 1$, $\chi = \pi/3$, $h = \sqrt{3}/2$ and $\zeta = \delta = 0$ at $t = 0$, which gives

$$u_n = -\sec(\frac{\pi n}{3})e^{i\kappa n}, \quad v_n = \sec(\frac{\pi n}{3})e^{-i\kappa n}. \quad (12)$$

Then $|u_n|^2 = |v_n|^2(n = 0,1,\ldots) = (1,4,4,1,4,4,1,4,4,1,\ldots)$ and the period is 3.

2.3. Dark Soliton

If $d_1 = d_3 = 0$, $d_0 = d_2^2/(4d_4)$, we possess dark soliton solutions ($p_s = 0$)

$$\begin{aligned} u_n &= a_1\sqrt{-\frac{d_2}{2d_4}}\tanh(\sqrt{-\frac{d_2}{2}}\xi)\exp(i\theta), \\ v_n &= b_1\sqrt{-\frac{d_2}{2d_4}}\tanh(\sqrt{-\frac{d_2}{2}}\xi)\exp(-i\theta), \end{aligned} \quad (13)$$

where functions ξ and θ are defined by (4) with

$$\varphi = \sqrt{-\frac{d_2}{2d_4}}\tanh(\sqrt{-\frac{d_2}{2}}\xi), \quad d_2 < 0, \quad d_4 > 0,$$

$$b_1 = \text{arbitrary constant}, \quad a_1 = \frac{2d_4\tanh^2(\sqrt{-\frac{d_2}{2}}\chi)}{b_1h^2d_2},$$

$$c = \frac{2\tanh(\sqrt{-\frac{d_2}{2}}k)\sin(\kappa)\sqrt{-2d_2}}{d_2h^2},$$

$$\omega = \frac{2\cos(\kappa)\text{sech}^2(\sqrt{-\frac{d_2}{2}}\chi) - 2}{h^2}.$$

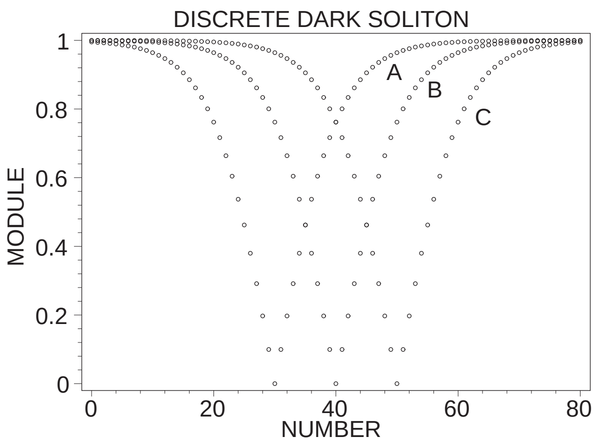

Fig. 3. Shape of discrete dark soliton for $|u_n|(|v_n|)$ with different phase ζ and parameters $d_2 = -d_4 = -1/50$, $b_1 = 1$, $h = \tanh(1/10)$, $\chi = 1$ at $t = 0$, where $\zeta = -30$ for A, $\zeta = -40$ for B, $\zeta = -50$ for C.

Similar to the discrete bright soliton, the parameter $\zeta(\delta)$ produces a continuous family of dark solitons with variable shape, see Figure 3. In the limit of large n, the solutions (13) reduce to constants

$$\begin{aligned} u_n &= \frac{d_4\sqrt{-\frac{2d_2}{d_4}}\tanh^2(\sqrt{-\frac{d_2}{2}}\chi)}{b_1h^2d_2}, \\ v_n &= b_1\sqrt{-\frac{d_2}{2d_4}}. \end{aligned} \quad (14)$$

These can be considered as another independent ('plane wave') solution of the AL system.

2.4. Oscillatory 'tan' Solution

The above dark soliton solutions also have extended forms involving trigonometric functions instead of the hyperbolic ones. The solutions ($p_s = 0$) are

$$\begin{aligned} u_n &= a_1\sqrt{\frac{d_2}{d_4}}\tan(\sqrt{\frac{d_2}{2}}\xi)\exp(i\theta), \\ v_n &= b_1\sqrt{\frac{d_2}{d_4}}\tan(\sqrt{\frac{d_2}{2}}\xi)\exp(-i\theta), \end{aligned} \quad (15)$$

where functions ξ and θ are defined by (4) with

$$\varphi = \sqrt{\frac{d_2}{d_4}}\tan(\sqrt{\frac{d_2}{2}}\xi), \quad d_2 > 0, \quad d_4 > 0,$$

$$b_1 = \text{arbitrary constant}, \quad a_1 = -\frac{d_4\tan^2(\sqrt{\frac{d_2}{2}}\chi)}{h^2b_1d_2},$$

$$c = -\frac{2\sin(\kappa)\tan(\sqrt{\frac{d_2}{2}}\chi)}{h^2\sqrt{d_2}},$$

$$\omega = \frac{2\cos(\kappa) - 2\cos^2(\sqrt{\frac{d_2}{2}}\chi)}{h^2\cos^2(\sqrt{\frac{d_2}{2}}\chi)}.$$

An example is $b_1 = d_2 = d_4 = 1$, $\chi = \sqrt{2}\pi/3$, $h = \sqrt{3}$ and $\zeta = \delta = 0$ at $t = 0$, which gives

$$u_n = -\tan(\frac{\pi n}{3})e^{i\kappa n}, \quad v_n = \tan(\frac{\pi n}{3})e^{-i\kappa n}. \quad (16)$$

Then $|u_n|^2 = |v_n|^2(n = 0,1,\ldots) = (0,3,3,0,3,3,0,3,3,0,\ldots)$ and the period is 3.

3. Elliptic Function Solutions

In this section, based on the extended Jacobian elliptic function algorithm for nonlinear differential equations presented by Zhengya Yan [29], we develop this method to the AL system (2). Though the modification is slight, it is important. In what follows we would like to simply introduce the extended Jacobian elliptic function expansion method.

Step 1: For a given system of M polynomial differential difference equations

$$\begin{aligned}&\Delta(u_{n+p_1}(x),\ldots,u_{n+p_\tau}(x),\ldots,u'_{n+p_1}(x),\\&\ldots,u'_{n+p_\tau}(x),\ldots,u^{(r)}_{n+p_1}(x),\ldots,u^{(r)}_{n+p_\tau}(x))=0,\end{aligned}\tag{17}$$

where the dependent variable u has M components u_i, the continuous variable x has N components x_i, the discrete variable n has Q components n_j, the τ shift vectors p_i, and $u^{(r)}(x)$ denotes the collection of mixed derivative terms of order r. When we seek the travelling wave solutions of (17), the first step is to introduce the wave transformation $u_{n+p_s}(x)=\varphi_{n+p_s}(\xi)$, $\xi=\sum_{i=1}^{Q}\chi_i n_i+\sum_{j=1}^{N}c_j x_j+\zeta$ for any s ($s=1,\ldots,\tau$), where the coefficients $c_1,c_2,\ldots,c_N,d_1,d_2,\ldots,d_Q$ and the phase ζ are all constants. In this way, (17) becomes

$$\begin{aligned}&\Delta(U_{n+p_1}(\xi),\ldots,U_{n+p_\tau}(\xi),\ldots,U'_{n+p_1}(\xi),\\&\ldots,U'_{n+p_\tau}(\xi),\ldots,U^{(r)}_{n+p_1}(\xi),\ldots,U^{(r)}_{n+p_\tau}(\xi))=0.\end{aligned}\tag{18}$$

Step 2: To seek the doubly periodic solutions of (17), we assume that (18) has the solution

$$\begin{aligned}&\varphi_{n+p_s}(\xi)=\sum_{j=1}^{l}a_jF^j_{n+p_s,i}(\xi),\\&i=1,2,\ldots,12,\quad s=1,2,\ldots,\tau\end{aligned}\tag{19}$$

with $F_{n+p_s,i}$ satisfying

$$\begin{aligned}&F_{n+p_s,1}(\xi)=\mathrm{sn}(\varsigma_n),\quad F_{n+p_s,2}(\xi)=\mathrm{cn}(\varsigma_n),\\&F_{n+p_s,3}(\xi)=\mathrm{dn}(\varsigma_n),\quad F_{n+p_s,4}(\xi)=\mathrm{ns}(\varsigma_n),\\&F_{n+p_s,5}(\xi)=\mathrm{nc}(\varsigma_n),\quad F_{n+p_s,6}(\xi)=\mathrm{nd}(\varsigma_n),\\&F_{n+p_s,7}(\xi)=\mathrm{cs}(\varsigma_n),\quad F_{n+p_s,8}(\xi)=\mathrm{ds}(\varsigma_n),\\&F_{n+p_s,9}(\xi)=\mathrm{cd}(\varsigma_n),\quad F_{n+p_s,10}(\xi)=\mathrm{sc}(\varsigma_n),\\&F_{n+p_s,11}(\xi)=\mathrm{sd}(\varsigma_n),\quad F_{n+p_s,12}(\xi)=\mathrm{dc}(\varsigma_n),\end{aligned}\tag{20}$$

and

$$\varsigma_n=\xi+v_s,\quad v_s=p_{s1}d_1+p_{s2}d_2+\cdots+p_{sQ}d_Q,\tag{21}$$

where l is the integer to be determined later. Moreover we know that $\mathrm{sn}(\varsigma_n)=\mathrm{sn}(\varsigma_n,m),\mathrm{cn}(\varsigma_n)=\mathrm{cn}(\varsigma_n,m)$ and $\mathrm{dn}(\varsigma_n)=\mathrm{dn}(\varsigma_n,m)$ of modulus m $(0\le m\le 1)$ are the Jacobian elliptic sine function, cosine function and the third kind Jacobian elliptic function. They are periodic with period $2K(m)$, where $K(m)$, $K(m)=\int_0^{\pi/2}\mathrm{d}x/\sqrt{1-m^2\sin^2 x}$, is the complete elliptic integral of the first kind. And other Jacobian functions are generated by these three kinds of functions, namely

$$\begin{aligned}&\mathrm{cs}(\varsigma_n)=\frac{\mathrm{cn}(\varsigma_n)}{\mathrm{sn}(\varsigma_n)},\ \mathrm{ds}(\varsigma_n)=\frac{\mathrm{dn}(\varsigma_n)}{\mathrm{sn}(\varsigma_n)},\ \mathrm{cd}(\varsigma_n)=\frac{\mathrm{cn}(\varsigma_n)}{\mathrm{dn}(\varsigma_n)},\\&\mathrm{sc}(\varsigma_n)=\frac{\mathrm{sn}(\varsigma_n)}{\mathrm{cn}(\varsigma_n)},\ \mathrm{sd}(\varsigma_n)=\frac{\mathrm{sn}(\varsigma_n)}{\mathrm{dn}(\varsigma_n)},\ \mathrm{dc}(\varsigma_n)=\frac{\mathrm{dn}(\varsigma_n)}{\mathrm{cn}(\varsigma_n)},\\&\mathrm{ns}(\varsigma_n)=\frac{1}{\mathrm{sn}(\varsigma_n)},\ \mathrm{nc}(\varsigma_n)=\frac{1}{\mathrm{cn}(\varsigma_n)},\ \mathrm{nd}(\varsigma_n)=\frac{1}{\mathrm{dn}(\varsigma_n)}.\end{aligned}\tag{22}$$

Yan [29] has shown their close relations and first-order derivatives.

Step 3: Determination of the truncation expansion terms in (19). l is fixed by balancing the linear term of the highest order with the highest nonlinear term in (18). Suppose we are interested in balancing terms with shift p_h, then terms with shifts other than p_h, say p_s, will not affect the balance since $F_{n+p_s,i}$ can be interpreted as being of degree zero in $F_{n+p_h,i}$.

Step 4: Substituting the ansatz (19) along with (20) and (21) into (18), then setting the coefficients of all powers like Jacobian elliptic functions to zero, we will get a series of algebraic equations with respect to the unknowns. Through solving the system of nonlinear algebraic equations we may determine these unknowns.

Step 5: By using the results obtained in the above steps we can derive many periodic solutions of (17). In addition we see that when $m\to 1$, $\mathrm{sn}(\varsigma_n)$, $\mathrm{cn}(\varsigma_n)$ and $\mathrm{dn}(\varsigma_n)$ degenerate as $\tanh(\varsigma_n)$, $\mathrm{sech}(\varsigma_n)$ and $\mathrm{sech}(\varsigma_n)$, respectively, while when $m\to 0$, $\mathrm{sn}(\varsigma_n)$, $\mathrm{cn}(\varsigma_n)$ and $\mathrm{dn}(\varsigma_n)$ degenerate as $\sin(\varsigma_n)$, $\cos(\varsigma_n)$ and 1, respectively. Therefore (19) degenerates as the solitonic solutions and trigonometric function solutions of (17).

Next, we would like to use the extended Jacobian elliptic function expansion method for the AL system (2). Apparently, this system has the continuous variable t and the discrete variable n, and $Q=N=1$ in step 1. Therefore, the AL system (2) also has the solutions expressed by (3) with (4), then (5) obtained.

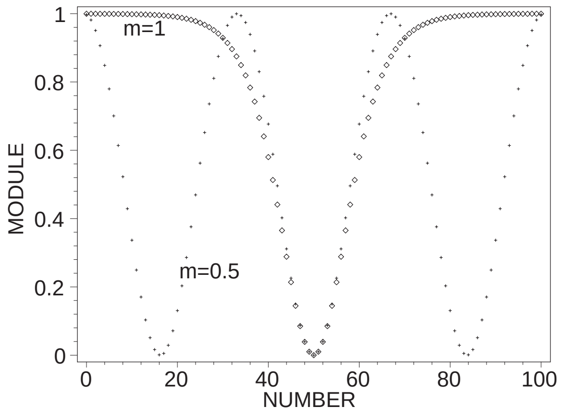

Fig. 4. Shape of discrete Jacobian sine periodic solution for $|u_n|^2(|v_n|^2)$ with parameters $b_1=-1$, $h=m\mathrm{sn}(\chi,m)$, $\chi=1/10$, $\zeta=-5$ at $t=0$.

According to the steps 2 and 3, we assume that (5) has the solution in the form

$$\phi_{n+p_s}(\xi)=\sum_{j=1}^{l_1} a_j F_{n+p_s,i}(\xi),\qquad \psi_{n+p_s}(\xi)=\sum_{j=1}^{l_2} b_j F_{n+p_s,i}(\xi), \tag{23}$$

where $F_{n+p_s,i}$, $(i=1,\ldots,12)$ satisfy (20) and function ξ defined by (4). Balancing the linear term of the highest order with the highest nonlinear term in (5), we have $l_1=l_2=1$. In what follows we shall discuss these cases, respectively.

sn(ς_n) expansion: When $i=1$ in ansatz (19), we have

$$\begin{aligned}&\phi_n(\xi)=a_1F_{n,1}(\xi),\qquad \psi_n(\xi)=b_1F_{n,1}(\xi),\\&\phi_{n+1}(\xi)=a_1F_{n+1,1}(\xi),\ \psi_{n+1}(\xi)=b_1F_{n+1,1}(\xi),\\&\phi_{n-1}(\xi)=a_1F_{n-1,1}(\xi),\ \psi_{n-1}(\xi)=b_1F_{n-1,1}(\xi),\end{aligned} \tag{24}$$

with

$$\begin{aligned}F_{n,1}(\xi)&=\mathrm{sn}(\xi),\\F_{n+1,1}(\xi)&=\mathrm{sn}(\xi+\chi)\\&=\frac{\mathrm{sn}(\xi)\mathrm{cn}(\chi)\mathrm{dn}(\chi)+\mathrm{sn}(\chi)\mathrm{cn}(\xi)\mathrm{dn}(\xi)}{1-m^2\mathrm{sn}^2(\xi)\mathrm{sn}^2(\chi)},\\F_{n-1,1}(\xi)&=\mathrm{sn}(\xi-\chi)\\&=\frac{\mathrm{sn}(\xi)\mathrm{cn}(\chi)\mathrm{dn}(\chi)-\mathrm{sn}(\chi)\mathrm{cn}(\xi)\mathrm{dn}(\xi)}{1+m^2\mathrm{sn}^2(\xi)\mathrm{sn}^2(\chi)}.\end{aligned} \tag{25}$$

Substituting expressions (24) and (25) into (5) by using the symbolic computation and according to the relations [11] yields a system of equations of $\mathrm{sn}^j(\xi)$ $(j=0,1,2)$. Setting their coefficients to zero yields a system of algebraic equations unknowns a_1, b_1, c and ω. By solving the system of algebraic equations, we can determines these parameters. Therefore we can obtain a Jacobian elliptic function solution of system (2) as follows

$$u_{1,n}=a_1\mathrm{sn}(\xi)\exp(\mathrm{i}\theta),\ v_{1,n}=b_1\mathrm{sn}(\xi)\exp(-\mathrm{i}\theta), \tag{26}$$

where functions ξ and θ are defined by (4) with

$$a_1=-\frac{m^2\mathrm{sn}^2(\chi)}{b_1h^2},\quad \omega=\frac{2\cos(\kappa)\mathrm{cn}(\chi)\mathrm{dn}(\chi)-2}{h^2},\quad c=-\frac{2\sin(\kappa)\mathrm{sn}(\chi)}{h^2},$$

where b_1, χ, κ and h are arbitrary constants. As an example we take $b_1=-1$, $h=m\mathrm{sn}(\chi,m)$, $\chi=2K(m)/3$, $m=0.8$ and $\zeta=\delta=0$, at $t=0$ in (26), which gives

$$u_n=\mathrm{sn}(\frac{2Kn}{3},0.8)e^{\mathrm{i}\kappa n},\quad v_n=-\mathrm{sn}(\frac{2Kn}{3},0.8)e^{-\mathrm{i}\kappa n}. \tag{27}$$

Then $|u_n|^2=|v_n|^2(n=0,1,\ldots)=(0,0.8356,\ 0.8356,0,0.8356,0.8356,0,0.8356,0.8356,0,\ldots)$ and the period is 3.

In particular when $m\to 1$, we get from (26) the kink-shaped soliton

$$\begin{aligned}u'_{1,n}&=-\frac{\tanh^2(\chi)}{c_1h^2}\tanh(\chi n-\frac{2\sin(\kappa)\tanh(\chi)}{h^2}t+\zeta)\exp[\mathrm{i}(\kappa n+\frac{2\cos(\kappa)\mathrm{sech}^2(\chi)-2}{h^2}t+\delta)],\\v'_{1,n}&=b_1\tanh(\chi n-\frac{2\sin(\kappa)\tanh(\chi)}{h^2}t+\zeta)\exp[-\mathrm{i}(\kappa n+\frac{2\cos(\kappa)\mathrm{sech}^2(\chi)-2}{h^2}t+\delta)].\end{aligned} \tag{28}$$

Figure 4 gives the shape of discrete Jacobian sine period solution for $|u_n|^2$ $(|v_n|^2)$ with different moduli m.

cn(ς_n) expansion: When $i = 2$, we have the solution of system (2)

$$\begin{aligned} u_{2,n} &= a_1 \mathrm{cn}(\chi n - \frac{2\sin(\kappa)\mathrm{sd}(\chi)}{h^2} t + \zeta)\exp[\mathrm{i}(\kappa n - \frac{2\mathrm{dn}^2(\chi) - 2\cos(\kappa)\mathrm{cn}(\chi)}{\mathrm{dn}^2(\chi) h^2} t + \delta)], \\ v_{2,n} &= \frac{m^2 \mathrm{sd}^2(\chi)}{a_1 h^2} \mathrm{cn}(\chi n - \frac{2\sin(\kappa)\mathrm{sd}(\chi)}{h^2} t + \zeta)\exp[-\mathrm{i}(\kappa n - \frac{2\mathrm{dn}^2(\chi) - 2\cos(\kappa)\mathrm{cn}(\chi)}{\mathrm{dn}^2(\chi) h^2} t + \delta)], \end{aligned} \tag{29}$$

where a_1, χ, κ and h are arbitrary constants.

Figure 5 gives the shape of discrete Jacobian cosine periodic solution for $|u_n|^2$ ($|v_n|^2$) with different moduli m. In particular when $m \to 1$, we get from (29) the bell-shaped soliton

$$\begin{aligned} u'_{2,n} &= a_1 \mathrm{sech}(\chi n - \frac{2\sin(\kappa)\sinh(\chi)}{\mathrm{sech}(\chi) h^2} t + \zeta)\exp[\mathrm{i}(\kappa n - \frac{2\mathrm{sech}(\chi) - 2\cos(\kappa)}{\mathrm{sech}(\chi) h^2} t + \delta)], \\ v'_{2,n} &= \frac{\sinh^2(\chi)}{b_1 h^2} \mathrm{sech}(\chi n - \frac{2\sin(\kappa)\sinh(\chi)}{h^2} t + \zeta)\exp[-\mathrm{i}(\kappa n - \frac{2\mathrm{sech}(\chi) - 2\cos(\kappa)}{\mathrm{sech}(\chi) h^2} t + \delta)]. \end{aligned} \tag{30}$$

dn(ς_n) expansion: When $i = 3$, we have the solution of system (2)

$$\begin{aligned} u_{3,n} &= \frac{\mathrm{sc}^2(\chi)}{b_1 h^2} \mathrm{dn}(\chi n - \frac{2\mathrm{sc}(\chi)\sin(\kappa)}{h^2} t + \zeta)\exp[\mathrm{i}(\kappa n + \frac{2\cos(\kappa)\mathrm{dn}(\chi) - 2\mathrm{cn}^2(\chi)}{\mathrm{cn}^2(\chi) h^2} t + \delta)], \\ v_{3,n} &= b_1 \mathrm{dn}(\chi n - \frac{2\mathrm{sc}(\chi)\sin(\kappa)}{h^2} t + \zeta)\exp[-\mathrm{i}(\kappa n + \frac{2\cos(\kappa)\mathrm{dn}(\chi) - 2\mathrm{cn}^2(\chi)}{\mathrm{cn}^2(\chi) h^2} t + \delta)], \end{aligned} \tag{31}$$

where b_1, χ, κ and h are arbitrary constants.

ns(ς_n) expansion: When $i = 4$, we have the solution of system (2)

$$\begin{aligned} u_{4,n} &= -\frac{\mathrm{sn}^2(\chi)}{b_1 h^2} \mathrm{ns}(\chi n - \frac{2\sin(\kappa)\mathrm{sn}(\chi)}{h^2} t + \zeta)\exp[\mathrm{i}(\kappa n + \frac{2\cos(\kappa)\mathrm{cn}(\chi)\mathrm{dn}(\chi) - 2}{h^2} t + \delta)], \\ v_{4,n} &= b_1 \mathrm{ns}(\chi n - \frac{2\sin(\kappa)\mathrm{sn}(\chi)}{h^2} t + \zeta)\exp[-\mathrm{i}(\kappa n + \frac{2\cos(\kappa)\mathrm{cn}(\chi)\mathrm{dn}(\chi) - 2}{h^2} t + \delta)], \end{aligned} \tag{32}$$

where b_1, χ, κ and h are arbitrary constants.

nc(ς_n) expansion: When $i = 5$, we have the solution of system (2)

$$\begin{aligned} u_{5,n} &= \frac{\mathrm{sd}^2(\chi)(m^2 - 1)}{b_1 h^2} \mathrm{nc}(\chi n - \frac{2\sin(\kappa)\mathrm{sd}(\chi)}{h^2} t + \zeta)\exp[\mathrm{i}(\kappa n + \frac{2\cos(\kappa)\mathrm{cn}(\chi) - 2\mathrm{dn}^2(\chi)}{h^2 \mathrm{dn}^2(\chi)} t + \delta)], \\ v_{5,n} &= b_1 \mathrm{nc}(\chi n - \frac{2\sin(\kappa)\mathrm{sd}(\chi)}{h^2} t + \zeta)\exp[-\mathrm{i}(\kappa n + \frac{2\cos(\kappa)\mathrm{cn}(\chi) - 2\mathrm{dn}^2(\chi)}{h^2 \mathrm{dn}^2(\chi)} t + \delta)], \end{aligned} \tag{33}$$

where b_1, χ, κ and h are arbitrary constants.

nd(ς_n) expansion: When $i = 6$, we have the solution of system (2)

$$\begin{aligned} u_{6,n} &= \frac{\mathrm{sc}^2(\chi)(1 - m^2)}{b_1 h^2} \mathrm{nd}(\chi n - \frac{2\sin(\kappa)\mathrm{sc}(\chi)}{h^2} t + \zeta)\exp[\mathrm{i}(\kappa n + \frac{2\cos(\kappa)\mathrm{dn}(\chi) - 2\mathrm{cn}^2(\chi)}{h^2 \mathrm{cn}^2(k)} t + \delta)], \\ v_{6,n} &= b_1 \mathrm{nd}(\chi n - \frac{2\sin(\kappa)\mathrm{sc}(\chi)}{h^2} t + \zeta)\exp[-\mathrm{i}(\kappa n + \frac{2\cos(\kappa)\mathrm{dn}(\chi) - 2\mathrm{cn}^2(\chi)}{h^2 \mathrm{cn}^2(\chi)} t + \delta)], \end{aligned} \tag{34}$$

where b_1, χ, κ and h are arbitrary constants.

cs(ς_n) expansion: When $i = 7$, we have the solution of system (2)

$$\begin{aligned} u_{7,n} &= -\frac{\mathrm{sc}^2(\chi)}{b_1 h^2}\mathrm{cs}\Big(\chi n - \frac{2\mathrm{sc}(\chi)\sin(\kappa)}{h^2}t + \zeta\Big)\exp\Big[\mathrm{i}\Big(\kappa n + \frac{2\cos(\kappa)\mathrm{dn}(\chi) - 2\mathrm{cn}^2(\chi)}{\mathrm{cn}^2(\chi)h^2}t + \delta\Big)\Big], \\ v_{7,n} &= b_1 \mathrm{cs}\Big(\chi n - \frac{2\mathrm{sc}(\chi)\sin(\kappa)}{h^2}t + \zeta\Big)\exp\Big[-\mathrm{i}\Big(\kappa n + \frac{2\cos(\kappa)\mathrm{dn}(\chi) - 2\mathrm{cn}^2(\chi)}{\mathrm{cn}^2(\chi)h^2}t + \delta\Big)\Big], \end{aligned} \tag{35}$$

where b_1, χ, κ and h are arbitrary constants.

ds(ς_n) expansion: When $i = 8$, we have the solution of system (2)

$$\begin{aligned} u_{8,n} &= -\frac{\mathrm{sd}^2(\chi)}{b_1 h^2}\mathrm{ds}\Big(\chi n - \frac{2\mathrm{sd}(\chi)\sin(\kappa)}{h^2}t + \zeta\Big)\exp\Big[\mathrm{i}\Big(\kappa n + \frac{2\cos(\kappa)\mathrm{cn}(\chi) - 2\mathrm{dn}^2(\chi)}{\mathrm{dn}^2(\chi)h^2}t + \delta\Big)\Big], \\ v_{8,n} &= b_1 \mathrm{ds}\Big(\chi n - \frac{2\mathrm{sd}(\chi)\sin(\kappa)}{h^2}t + \zeta\Big)\exp\Big[-\mathrm{i}\Big(\kappa n + \frac{2\cos(\kappa)\mathrm{cn}(\chi) - 2\mathrm{dn}^2(\chi)}{\mathrm{dn}^2(\chi)h^2}t + \delta\Big)\Big], \end{aligned} \tag{36}$$

where b_1, χ, κ and h are arbitrary constants.

cd(ς_n) expansion: When $i = 9$, we have the solution of system (2)

$$\begin{aligned} u_{9,n} &= -\frac{\mathrm{sn}^2(\chi)m^2}{b_1 h^2}\mathrm{cd}\Big(\chi n - \frac{2\sin(\kappa)\mathrm{sn}(\chi)}{h^2}t + \zeta\Big)\exp\Big[\mathrm{i}\Big(\kappa n + \frac{-2 + 2\cos(\kappa)\mathrm{dn}(\chi)\mathrm{cn}(\chi)}{h^2}t + \delta\Big)\Big], \\ v_{9,n} &= b_1 \mathrm{cd}\Big(\chi n - \frac{2\sin(\kappa)\mathrm{sn}(\chi)}{h^2}t + \zeta\Big)\exp\Big[-\mathrm{i}\Big(\kappa n + \frac{-2 + 2\cos(\kappa)\mathrm{dn}(\chi)\mathrm{cn}(\chi)}{h^2}t + \delta\Big)\Big], \end{aligned} \tag{37}$$

where b_1, χ, κ and h are arbitrary constants.

sc(ς_n) expansion: When $i = 10$, we have the solution of system (2)

$$\begin{aligned} u_{10,n} &= \frac{\mathrm{sc}^2(\chi)(m^2 - 1)}{b_1 h^2}\mathrm{sc}\Big(\chi n - \frac{2\sin(\kappa)\mathrm{sc}(\chi)}{h^2}t + \zeta\Big)\exp\Big[\mathrm{i}\Big(\kappa n + \frac{2\cos(\kappa)\mathrm{dn}(\chi) - 2\mathrm{cn}^2(\chi)}{h^2\mathrm{cn}^2(\chi)}t + \delta\Big)\Big], \\ v_{10,n} &= b_1 \mathrm{sc}\Big(\chi n - \frac{2\sin(\kappa)\mathrm{sc}(\chi)}{h^2}t + \zeta\Big)\exp\Big[-\mathrm{i}\Big(\kappa n + \frac{2\cos(\kappa)\mathrm{dn}(k) - 2\mathrm{cn}^2(\chi)}{h^2\mathrm{cn}^2(\chi)}t + \delta\Big)\Big], \end{aligned} \tag{38}$$

where b_1, χ, κ and h are arbitrary constants.

sd(ς_n) expansion: When $i = 11$, we have the solution of system (2)

$$\begin{aligned} u_{11,n} &= -\frac{m^2\mathrm{sd}^2(\chi)(m^2 - 1)}{b_1 h^2}\mathrm{sd}\Big(\chi n - \frac{2\sin(\kappa)\mathrm{sd}(\chi)}{h^2}t + \zeta\Big)\exp\Big[\mathrm{i}\Big(\kappa n + \frac{2\cos(\kappa)\mathrm{cn}(\chi) - 2\mathrm{dn}^2(\chi)}{h^2\mathrm{dn}^2(\chi)}t + \delta\Big)\Big], \\ v_{11,n} &= b_1 \mathrm{sd}\Big(\chi n - \frac{2\sin(\kappa)\mathrm{sd}(\chi)}{h^2}t + \zeta\Big)\exp\Big[-\mathrm{i}\Big(\kappa n + \frac{2\cos(\kappa)\mathrm{cn}(\chi) - 2\mathrm{dn}^2(\chi)}{h^2\mathrm{dn}^2(\chi)}t + \delta\Big)\Big], \end{aligned} \tag{39}$$

where b_1, χ, κ and h are arbitrary constants.

dc(ς_n) expansion: When $i = 12$, we have the solution of system (2)

$$\begin{aligned} u_{12,n} &= -\frac{\mathrm{sn}^2(\chi)}{b_1 h^2}\mathrm{dc}\Big(\chi n - \frac{2\sin(\kappa)\mathrm{sn}(\chi)}{h^2}t + \zeta\Big)\exp\Big[\mathrm{i}\Big(\kappa n + \frac{-2 + 2\cos(\kappa)\mathrm{dn}(\chi)\mathrm{cn}(\chi)}{h^2}t + \delta\Big)\Big], \\ v_{12,n} &= b_1 \mathrm{dc}\Big(\chi n - \frac{2\sin(\kappa)\mathrm{sn}(\chi)}{h^2}t + \zeta\Big)\exp\Big[-\mathrm{i}\Big(\kappa n + \frac{-2 + 2\cos(\kappa)\mathrm{dn}(\chi)\mathrm{cn}(\chi)}{h^2}t + \delta\Big)\Big], \end{aligned} \tag{40}$$

where b_1, χ, κ and h are arbitrary constants.

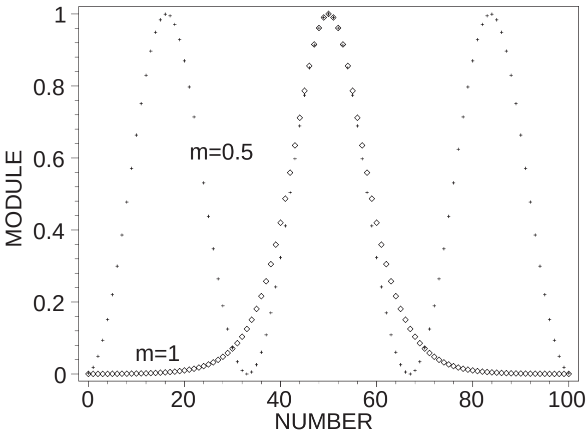

Fig. 5. Shape of discrete Jacobian cosine periodic solution for $|u_n|^2$ ($|v_n|^2$) with parameters $a_1 = 1$, $h = m\mathrm{sd}(\chi, m)$, $\chi = 1/10$, $\zeta = -5$ at $t = 0$.

4. Summary

We have analyzed the AL system, and have found a set of solutions including elliptic Jacobian doubly periodic solutions, bright and dark soliton solutions, and quasi-periodic solutions. The width and amplitude of soliton solutions are fixed by some arbitrary parameters. The lattice admits ‘sec’-type solutions without the singularities characteristic of continuous models, where the amplitude of such solutions can go to infinity at certain points. In the case lattice, the points of singularity can be ’between sits’ and hence the singularities can be avoided. Finding exact solutions is an important step in studying discrete lattices. We cannot claim, though, that our solutions exhaust the list of possible solutions. Further work is needed to find other possible types of solutions.

Another issue is the stability of the solutions. Similar to the other integrable systems, it is very likely that the exact solutions of the AL system in this paper are stable. For instance, AL system (2) has a two-parameter exact soliton solution, namely

$$u_n(t) = \frac{\sinh(\chi)\exp[\mathrm{i}\kappa(n-n_0)+\mathrm{i}\omega t]}{\cosh[\chi(n-vt-n_0)]},$$
$$v_n(t) = \frac{\sinh(\chi)\exp[-\mathrm{i}\kappa(n-n_0)-\mathrm{i}\omega t]}{\cosh[\chi(n-vt-n_0)]},$$

i. e. taking $a_1 = \sinh(\chi)$, $d_2 = -d_4 = h = 1$, $c = -\chi v$, $\zeta = -\chi n_0$, $\delta = -\kappa n_0$ in (10), where χ and κ are free parameters of the solution, $\omega = -2+2\cos(\kappa)\cosh(k)$, and the velocity is given by

$$v = \frac{2\sinh(\chi)\sin(\kappa)}{\chi}.$$

Some remarkable properties of this solutions are as follows:

1.) the solution has translational invariance, since n_0 is an arbitrary real number, although the lattice itself does not have this invariance;

2.) the solution has an arbitrary amplitude, defined by the parameter χ;

3.) the solution can move with arbitrary velocity v.

All the above properties are consequences of the integrability [27]. The solutions of integrable models are usually neutrally stable.

Although it appears simple, the translational invariance is not as trivial as in the case of the corresponding continuous equation. When $\zeta = -\chi n_0$ is zero, the center of the soliton coincides with a lattice site. In this instance, the solution is symmetric. When ζ is not zero, the soliton center is located between two lattice sites. Then the soliton shape can be asymmetric. In this sense, the parameter ζ produces a continuous family of solitons with variable shape. Certainly, we can discuss the common features of the other exact solutions in this paper with the similar analyse.

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